55[F].—KENNETH I. APPEL & J. BARKLEY ROSSER, *Table for Estimating Functions of Primes*, Communications Research Division Technical Report No. 4, Institute for Defense Analyses, Princeton, N. J., 1961, xxxii + 125 p., 22 cm.

This interesting table is quite complicated in its design, and partly so in its subject matter. For the sake of brevity a complete description will not be given here. The functions of the primes referred to in the title are primarily $\pi(x)$, the number of primes $\leqq x$, the three sums over the primes p:

$$\sum_{p \leqq x} \log p, \qquad \sum_{p \leqq x} \frac{1}{p}, \qquad \sum_{p \leqq x} \frac{\log p}{p},$$

and the product

$$\prod_{p \leqq x} \frac{p}{p-1}.$$

The range of x is from 2 to 10^8.

The approach used here may be illustrated with the first sum:

$$\theta(x) = \sum_{p \leqq x} \log p.$$

It is well-known that $\theta(x) \sim x$, (this is equivalent to the Prime Number Theorem), and if one defines the coefficient $TH(x)$ by

$$\theta(x) = x - TH(x) \cdot \sqrt{x},$$

it is found *empirically* that $TH(x)$ has a rather small variation over the range x for which it has been computed. For example, between $x = 85{,}881{,}353$ and $x = 87{,}679{,}913$, $TH(x)$ has a maximum of 1.337 and a minimum of 1.015. Knowing these extremes, one could therefore obtain bounds for, say, $\theta(86{,}692{,}297)$, as follows:

$$86{,}679{,}848 \leqq \theta \leqq 86{,}682{,}847.$$

The exact value for this argument is $\theta = 86{,}681{,}759.3$.

Similarly, if one knows bounds on

$$PI(x) = [li(x) - \pi(x)] \log x \cdot x^{-1/2},$$

$$SR(x) = x^{1/2} \log x \left[\sum_{p \leqq x} \frac{1}{p} - \log \log x - 0.261497 \right],$$

and

$$SL(x) = x^{1/2} \left[\sum_{p \leqq x} \frac{\log p}{p} - \log x + 1.332582 \right],$$

one can obtain good estimates for $\pi(x)$ and for the second and third sums above.

Further, the authors also give more complicated, third-order correction coefficients which enable one to obtain even more accurate estimates.

The corresponding coefficient for the product:

$$PR(x) = x^{1/2}\left[e^{-\gamma} \prod_{p \leqq x} \frac{p}{p-1} - \log x\right],$$

is not listed as such, since the authors, by study of their preliminary computations, discovered the approximate formula:

$$PR(x) \approx SR(x) + \frac{(SR(x))^2}{2x^{1/2} \log x} - \frac{\log x}{2x^{1/2}(1 + \log x)}.$$

Professor Lowell Schoenfeld later obtained a rigorous justification. His long proof is given in full in the Introduction.

The five primary functions are listed, first, for each of the first 64 primes; $\pi(x)$ exactly, $\theta(x)$ to 5D, and the other three functions to 10D. The primes from 313 to 99,999,989 are divided into 173 rather irregularly spaced intervals, and in each interval the five functions are listed at every x for which one of the auxiliary functions, $TH(x)$, $PI(x)$, etc., takes on a maximum or a minimum value. These extrema of $TH(x)$, etc., are also given.

In addition, the largest gap between successive primes in each interval is listed. The largest gap up to 10^8 occurs between the primes 47,326,693 and 47,326,913.

The orientation here is that of estimating the primary functions in terms of the bounded coefficients. But from a theoretical point of view the opposite orientation is probably one of greater interest. One would like to know the order of the true bounds upon these coefficients. The $x^{1/2}$ that enters into all of their defining equations is related to the Riemann Hypothesis, and, as is well-known, the state of the theory here leaves much to be desired. It has been suggested, in an off-hand manner, in *MTAC*, v. 13, 1959, p. 282, that $PI(x)$ has a mean value equal to 1. The range of its values given here, for $313 \leqq x \leqq 10^8$, is

$$0.526 \leqq PI(x) \leqq 2.742.$$

Finally, a word concerning the table *per se*. Since the subject matter is so fundamental, an improved and more elegant edition is probably called for. While the table, as it stands, is quite workable, a less erratic selection of intervals and a somewhat clearer format would be desirable.

D. S.

56[G].—G. E. Shilov, *An Introduction to the Theory of Linear Spaces*, Prentice-Hall, Inc. New Jersey, 1961, ix + 310 p., 23 cm. Price $10.00.

This book is the first in Prentice-Hall's series of translations from the Russian. A bibliography has been added by the translator, R. A. Silverman.

The contents include the usual topics in linear algebra such as determinants, linear spaces, systems of linear equations, coordinate transformations, invariant subspaces and eigenvalues of linear transformations, and quadratic forms. The degree of abstraction is shown by the fact that sections on ideals and tensors are included, but marked with asterisks to indicate that they may be omitted if desired. The final chapter deals with infinite-dimensional Euclidean spaces.

The book is not concerned with computing methods directly, so that its value